On the Triangular Decomposition of Cauchy Matrices

By Choong Yun Cho

1. Introduction. Recently Schechter [4], and (apparently independently) Trench and Scheinok [5] have shown that the elements of the inverse of a Cauchy matrix may be written down in simple closed form.

In this short note we demonstrate that the triangular decomposition of the Cauchy matrix may also be written out explicitly, and furthermore that the same is true of the inverse of these triangular factors.

The inverses of the triangular factors are deduced from new determinantal expressions for the elements of the triangular factors of a general matrix.

2. Preliminary Results. It is convenient to use the following notation for the general square matrix A of order k:

$$A = (a_1 b_2 c_3 \cdots v_{k-1} w_k) = \begin{bmatrix} a_1 & b_1 & c_1 & \cdots & w_1 \\ a_2 & b_2 & c_2 & \cdots & w_2 \\ & & \cdot & & \\ & & \cdot & & \\ & & \cdot & & \\ a_k & b_k & c_k & \cdots & w_k \end{bmatrix}.$$

We denote the determinant of this matrix by $|a_1 b_2 c_3 \cdots w_k|$. Thus

$$|b_1 d_3 h_4| = \det \begin{bmatrix} b_1 & d_1 & h_1 \\ b_3 & d_3 & h_3 \\ b_4 & d_4 & h_4 \end{bmatrix}.$$

It is known (Turnbull [6, p. 369]) that for the usual triangular decomposition $A = LU$, the elements of L and U can be expressed in terms of determinants involving the elements of A as follows (it is assumed that the decomposition is possible and L has been chosen to have units in the principal diagonal):

$$(1) \qquad L = \begin{bmatrix} 1 & & & & & \\ \frac{|a_2|}{|a_1|} & 1 & & & & \\ \frac{|a_3|}{|a_1|} & \frac{|a_1 b_3|}{|a_1 b_2|} & 1 & & & \\ & \cdots & & & & \\ \frac{|a_m|}{|a_1|} & \frac{|a_1 b_m|}{|a_1 b_2|} & \frac{|a_1 b_2 c_m|}{|a_1 b_2 c_3|} & \cdots & 1 & \\ & \cdots & & & & \\ \frac{|a_k|}{|a_1|} & \frac{|a_1 b_k|}{|a_1 b_2|} & \frac{|a_1 b_2 c_k|}{|a_1 b_2 c_3|} & \frac{|a_1 b_2 c_3 d_k|}{|a_1 b_2 c_3 d_4|} & \cdots & 1 \end{bmatrix}$$

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and

$$(2)\quad U = \begin{bmatrix} |a_1| & |b_1| & |c_1| & |d_1| & \cdots & |w_1| \\ & \frac{|a_1b_2|}{|a_1|} & \frac{|a_1c_2|}{|a_1|} & \frac{|a_1d_2|}{|a_1|} & \cdots & \frac{|a_1w_2|}{|a_1|} \\ & & \frac{|a_1b_2c_3|}{|a_1b_2|} & \frac{|a_1b_2d_3|}{|a_1b_2|} & \cdots & \frac{|a_1b_2w_3|}{|a_1b_2|} \\ & & & & & \vdots \\ & & & & & \frac{|a_1b_2 \cdots v_{k-1}w_k|}{|a_1b_2 \cdots v_{k-1}|} \end{bmatrix}.$$

It does not seem to have been noticed that L^{-1} and U^{-1} can similarly be expressed explicitly as follows:

$$(3)\quad L^{-1} = \begin{bmatrix} 1 & & & \\ -\frac{|a_2|}{|a_1|} & 1 & & \\ \frac{|a_2b_3|}{|a_1b_2|} & -\frac{|a_1b_3|}{|a_1b_2|} & 1 & \\ -\frac{|a_2b_3c_4|}{|a_1b_2c_3|} & \frac{|a_1b_3c_4|}{|a_1b_2c_3|} & -\frac{|a_1b_2c_4|}{|a_1b_2c_3|} & 1 \\ & \vdots & & \\ (-1)^{k-1}\frac{|a_2b_3 \cdots v_k|}{|a_1b_2 \cdots v_{k-1}|} & (-1)^{k}\frac{|a_1b_3c_4 \cdots v_k|}{|a_1b_2c_3 \cdots v_{k-1}|} & \cdots & 1 \end{bmatrix}$$

and

$$(4)\quad U^{-1} = \begin{bmatrix} \frac{|1|}{|a_1|} & -\frac{|b_1|}{|a_1b_2|} & \frac{|b_1c_2|}{|a_1b_2c_3|} & -\frac{|b_1c_2d_3|}{|a_1b_2c_3d_4|} & \cdots & (-1)^{k-1}\frac{|b_1c_2d_3 \cdots w_{k-1}|}{|a_1b_2c_3 \cdots w_k|} \\ & \frac{|a_1|}{|a_1b_2|} & -\frac{|a_1c_2|}{|a_1b_2c_3|} & \frac{|a_1c_2d_3|}{|a_1b_2c_3d_4|} & \cdots & (-1)^{k}\frac{|a_1c_2d_3 \cdots w_{k-1}|}{|a_1b_2c_3 \cdots w_k|} \\ & & \frac{|a_1b_2|}{|a_1b_2c_3|} & -\frac{|a_1b_2d_3|}{|a_1b_2c_3d_4|} & \cdots & (-1)^{k+1}\frac{|a_1b_2d_3 \cdots w_{k-1}|}{|a_1b_2c_3 \cdots w_k|} \\ & & & & & \vdots \\ & & & & & \frac{|a_1b_2c_3 \cdots v_{k-1}|}{|a_1b_2c_3 \cdots w_k|} \end{bmatrix}.$$

These results can be shown as follows (the author is indebted to the referee for this simple and elegant proof):

Since $L^{-1}A = U$, the requirement that the product $L^{-1}A$ be upper triangular imposes $(m - 1)$ conditions on the $(m - 1)$ elements of L^{-1} in the mth row of $L^{-1}A$, i.e.,

$$(L^{-1}A)_{ms} = 0 \qquad (m > s)$$

or

$$\sum_{r=1}^{m-1} (L^{-1})_{mr} A_{rs} = -A_{ms}, \qquad s = 1, 2, \cdots, m-1 .$$

This is a system of $(m - 1)$ equations for the unknowns $(L^{-1})_{mr}$, $r = 1, 2, \cdots, m - 1$, and formula (3) for L^{-1} is merely Cramer's rule.

Similarly, we have

$$(AU^{-1})_{sm} = 0 \qquad (m > s)$$

or

$$\sum_{r=1}^{m-1} A_{sr}(U^{-1})_{rm} = -A_{sm}(U^{-1})_{mm}, \qquad s = 1, 2, \cdots, m-1 .$$

But (see [2, p. 11])

$$(U^{-1})_{mm} = \frac{|a_1 b_2 \cdots g_{m-1}|}{|a_1 b_2 \cdots g_{m-1} h_m|},$$

and by the same argument expression (4) for U^{-1} is obtained from Cramer's rule.

In some problems one wishes to decompose a matrix into a product of the form UL rather than in the more usual form LU. If now we begin with $A = UL$, where A is the general square matrix of order k, $A = (a_1 b_2 c_3 \cdots v_{k-1} w_k)$, then we obtain by the same method

$$(5) \quad U = \begin{bmatrix} \frac{|a_1b_2c_3d_4\cdots w_k|}{|b_2c_3d_4\cdots w_k|} & \frac{|b_1c_3d_4\cdots w_k|}{|c_3d_4\cdots w_k|} & \frac{|c_1d_4\cdots w_k|}{|d_4\cdots w_k|} & \cdots & \frac{|v_1w_k|}{|w_k|} & |w_1| \\ & \frac{|b_2c_3d_4\cdots w_k|}{|c_3d_4\cdots w_k|} & \frac{|c_2d_4\cdots w_k|}{|d_4\cdots w_k|} & \cdots & \frac{|v_2w_k|}{|w_k|} & |w_2| \\ & & \frac{|c_3d_4\cdots w_k|}{|d_4\cdots w_k|} & \cdots & \frac{|v_3w_k|}{|w_k|} & |w_3| \\ & & & & \vdots & \\ & & & & \frac{|v_{k-1}w_k|}{|w_k|} & |w_{k-1}| \\ & & & & & |w_k| \end{bmatrix}$$

and

$$
(6)\qquad L=\begin{bmatrix}
1 \\
\frac{|a_2c_3d_4e_5\cdots w_k|}{|b_2c_3d_4e_5\cdots w_k|} & 1 \\
\frac{|a_3d_4e_5\cdots w_k|}{|c_3d_4e_5\cdots w_k|} & \frac{|b_3d_4e_5\cdots w_k|}{|c_3d_4e_5\cdots w_k|} & 1 \\
\frac{|a_4e_5\cdots w_k|}{|d_4e_5\cdots w_k|} & \frac{|b_4e_5\cdots w_k|}{|d_4e_5\cdots w_k|} & \frac{|c_4e_5\cdots w_k|}{|d_4e_5\cdots w_k|} & 1 \\
 & \vdots \\
\frac{|a_k|}{|w_k|} & \frac{|b_k|}{|w_k|} & \frac{|c_k|}{|w_k|} & \cdots & 1
\end{bmatrix}.
$$

Their inverses are

$$
(7)\qquad U^{-1}=\begin{bmatrix}
\frac{|b_2c_3d_4\cdots w_k|}{|a_1b_2c_3d_4\cdots w_k|} & -\frac{|b_1c_3d_4\cdots w_k|}{|a_1b_2c_3d_4\cdots w_k|} & \frac{|b_1c_2d_4\cdots w_k|}{|a_1b_2c_3d_4\cdots w_k|} & \cdots & (-1)^{k-1}\frac{|b_1c_2d_3\cdots w_{k-1}|}{|a_1b_2c_3d_4\cdots w_k|} \\
 & \frac{|c_3d_4\cdots w_k|}{|b_2c_3d_4\cdots w_k|} & -\frac{|c_2d_4\cdots w_k|}{|b_2c_3d_4\cdots w_k|} & \cdots & (-1)^{k}\frac{|c_2d_3\cdots w_{k-1}|}{|b_2c_3d_4\cdots w_k|} \\
 & & \frac{|d_4\cdots w_k|}{|c_3d_4\cdots w_k|} & \cdots & (-1)^{k+1}\frac{|d_3\cdots w_{k-1}|}{|c_3d_4\cdots w_k|} \\
 & & & & \vdots \\
 & & & & \frac{|1|}{|w_k|}
\end{bmatrix}
$$

and

$$
(8)\qquad L^{-1}=\begin{bmatrix}
1 \\
-\frac{|a_2c_3d_4\cdots w_k|}{|b_2c_3d_4\cdots w_k|} & 1 \\
\frac{|a_2b_3d_4\cdots w_k|}{|b_2c_3d_4\cdots w_k|} & -\frac{|b_3d_4\cdots w_k|}{|c_3d_4\cdots w_k|} & 1 \\
-\frac{|a_2b_3c_4\cdots w_k|}{|b_2c_3d_4\cdots w_k|} & \frac{|b_3c_4\cdots w_k|}{|c_3d_4\cdots w_k|} & -\frac{|c_4\cdots w_k|}{|d_4\cdots w_k|} & 1 \\
 & \vdots \\
(-1)^{k-1}\frac{|a_2b_3c_4\cdots v_k|}{|b_2c_3d_4\cdots w_k|} & (-1)^{k}\frac{|b_3c_4\cdots v_k|}{|c_3d_4\cdots w_k|} & (-1)^{k+1}\frac{|c_4\cdots v_k|}{|d_4\cdots w_k|} & \cdots & 1
\end{bmatrix}.
$$

These expressions, Eq. (1) to Eq. (8) inclusively, constitute a complete set of the formulas for the triangular decomposition of a general square matrix, expressed in terms of determinants.

3. Main Results. We denote the Cauchy matrix of order k

$$C = \begin{bmatrix} \frac{1}{\alpha_1 + \beta_1} & \frac{1}{\alpha_1 + \beta_2} & \cdots & \frac{1}{\alpha_1 + \beta_k} \\ \frac{1}{\alpha_2 + \beta_1} & \frac{1}{\alpha_2 + \beta_2} & \cdots & \frac{1}{\alpha_2 + \beta_k} \\ & & \vdots & \\ \frac{1}{\alpha_k + \beta_1} & \frac{1}{\alpha_k + \beta_2} & \cdots & \frac{1}{\alpha_k + \beta_k} \end{bmatrix}$$

by $((1/(\alpha_m + \beta_n); m, n = 1, 2, \cdots, k))$.

THEOREM 1. *Given the Cauchy matrix of order* k, $C = ((1/(\alpha_m + \beta_n); m, n = 1, 2, \cdots, k))$, *where* $\alpha_1, \alpha_2, \cdots, \alpha_k; -\beta_1, -\beta_2, \cdots, -\beta_k$, *are pairwise distinct and* $C = LU$. *Then the triangular factors are*

$$L_{m,n} = \frac{(\alpha_n + \beta_n)}{(\alpha_m + \beta_n)} \cdot \prod_{s=1}^{n-1} \frac{(\alpha_n + \beta_s)(\alpha_m - \alpha_s)}{(\alpha_m + \beta_s)(\alpha_n - \alpha_s)}, \tag{9}$$

$$U_{m,n} = \frac{1}{(\alpha_m + \beta_n)} \cdot \prod_{s=1}^{m-1} \frac{(\alpha_m - \alpha_s)(\beta_n - \beta_s)}{(\alpha_s + \beta_n)(\alpha_m + \beta_s)}. \tag{10}$$

Proof. We merely have to substitute $(\alpha_m + \beta_n)^{-1}$ in Eq. (1) and observe that the mth row of L has elements formed by the quotients of two determinants. Each numerator is obtained by replacing the last row of the matrix whose determinant is in the denominator with the corresponding elements of the mth row of the original given matrix. Note that m is always larger than the order of the determinant under consideration.

Using the formula for the determinant of Cauchy form (see, e.g., Pólya and Szegö [3, p. 98]), a simple arithmetic manipulation yields formula (9). Formula (10) can be derived in an analogous fashion.

In a very similar way we prove

THEOREM 2. *The elements of the matrix inverses of* L *and* U, *where* $C = LU$, *are given by*

$$(L^{-1})_{m,n} = \prod_{t=1;\, t\neq n}^{m} \frac{1}{(\alpha_n - \alpha_t)} \cdot \prod_{s=1}^{m-1} \frac{(\alpha_m - \alpha_s)(\alpha_n + \beta_s)}{(\alpha_m + \beta_s)} \qquad (m \geqq n), \tag{11}$$

$$(U^{-1})_{m,n} = \frac{(\alpha_n + \beta_n)}{\prod_{t=1;\, t\neq m}^{n} (\beta_m - \beta_t)} \cdot \prod_{s=1}^{n-1} \frac{(\alpha_s + \beta_m)(\alpha_n + \beta_s)}{(\alpha_n - \alpha_s)} \qquad (m \leqq n). \tag{12}$$

THEOREM 3. *Given the Cauchy matrix* C *of order* k *as defined in Theorem* 1, *let* $C = UL$, *then*

$$U_{m,n} = \frac{1}{(\alpha_m + \beta_n)} \cdot \prod_{s=n+1}^{k} \frac{(\alpha_m - \alpha_s)(\beta_n - \beta_s)}{(\alpha_s + \beta_n)(\alpha_m + \beta_s)}, \tag{13}$$

$$(14)\qquad L_{m,n} = \frac{(\alpha_m + \beta_m)}{(\alpha_m + \beta_n)} \cdot \prod_{s=m+1}^{k} \frac{(\alpha_s + \beta_m)(\beta_n - \beta_s)}{(\alpha_s + \beta_n)(\beta_m - \beta_s)},$$

$$(15)\qquad (U^{-1})_{m,n} = \frac{(\alpha_m + \beta_m)}{\prod_{t=m;\, t\neq n}^{k} (\alpha_n - \alpha_t)} \cdot \prod_{s=m+1}^{k} \frac{(\alpha_n + \beta_s)(\alpha_s + \beta_m)}{(\beta_m - \beta_s)} \qquad (m \leqq n),$$

$$(16)\qquad (L^{-1})_{m,n} = \prod_{t=n;\, t\neq m}^{k} \frac{1}{(\beta_m - \beta_t)} \cdot \prod_{s=n+1}^{k} \frac{(\alpha_s + \beta_m)(\beta_n - \beta_s)}{(\alpha_s + \beta_n)} \qquad (m \geqq n).$$

The proof of this theorem can be shown by the same sort of elementary manipulations used to establish Theorems 1 and 2.

(*Remark*. All the empty products in above expressions are assumed to be unity.)

4. Hankel and Toeplitz Matrices. In two special cases it turns out that the Cauchy matrix reduces to one of Toeplitz form: These occur when

$$T_1 = ((1/(m - n + x); m, n = 1, 2, \cdots, k))$$

where x is a constant, and

$$T_2 = ((1/(p - q^{m-n+x}); m, n = 1, 2, \cdots, k))$$

where p, q, and x are constants.

In the first case we obtain readily L, U, L^{-1}, and U^{-1} by putting $\alpha_m = m$ and $\beta_n = x - n$. The inverse becomes

$$(17)\qquad (T_1^{-1})_{m,n} = \frac{\prod_{t=1}^{k} (x + t - m)(x - t + n)}{(n - m + x) \cdot \prod_{r=1;\, r\neq m}^{k} (m - r) \cdot \prod_{s=1;\, s\neq n}^{k} (s - n)}.$$

In the second case we note that

$$T_2 = ((1/(pq^n - q^{m+x}); m, n = 1, 2, \cdots, k)) \cdot D,$$

where D is a diagonal matrix with $d_{nn} = q^n$, $n = 1, 2, \cdots, k$.

The first factor of T_2 now has the Cauchy form, i.e., $\alpha_m = -q^{m+x}$ and $\beta_n = pq^n$, and the triangular factors are readily expressed. The inverse of T_2 is

$$(18)\qquad (T_2^{-1})_{m,n} = \frac{\prod_{t=1}^{k} (pq^m - q^{x+t})(pq^t - q^{x+n})}{y \cdot \prod_{r=1;\, r\neq m}^{k} (q^r - q^m) \cdot \prod_{s=1;\, s\neq n}^{k} (q^n - q^s)}$$

where $y = p^k q^{kx-x+2m} - p^{k-1} q^{kx+m+n}$.

The Hankel forms corresponding to the above Toeplitz matrices T_1 and T_2 are

$$H_1 = ((1/(m + n + x); m, n = 1, 2, \cdots, k))$$

and

$$H_2 = ((1/(p - q^{m+n+x}); m, n = 1, 2, \cdots, k)),$$

respectively.

The triangular factors of H_1 and H_2, and their inverses are immediately obtained by choosing α_m and β_n in the theorems. In particular, the inverse of H_2 is

$$(H_2^{-1})_{m,n} = \frac{\prod_{t=1}^{k} (pq^{-m} - q^{x+t})(pq^{-t} - q^{x+n})}{y \cdot \prod_{r=1;\, r \neq m}^{k} (q^{-r} - q^{-m}) \cdot \prod_{s=1;\, s \neq n}^{k} (q^{n} - q^{s})} \tag{19}$$

where

$$y = p^k q^{kx-x-2m} - p^{k-1} q^{kx-m+n} .$$

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